臺灣綜合大學系統 109 學年度學士班轉學生聯合招生考試試題

科目名稱	線性代數	類組代碼	D25
		科目碼	D2592

※本項考試依簡章規定所有考科均「不可」使用計算機。　　本科試題共計 2 頁

1. (14 points)

   (a) (5 points) Find a 3×3 nonsingular matrix A satisfying $3A = A^2 + AB$, where

$$B = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

   (b) (9 points) Find the inverse of A.

2. (16 points) Let A be the matrix

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & -1 & 0 \\ 0 & -4 & 3 \end{bmatrix}$$

   (a) (8 points) Find the eigenvalues (and their algebraic and geometric multiplicities) and their eigenspaces.

   (b) (8 points) Find A^n, where n is a positive integer.

3. (10 points) Let X be the linear subspace of $\mathbb{R}^{2\times 3}$ containing all matrices whose columns add to $0 \in \mathbb{R}^2$. For instance, $\begin{pmatrix} 2 & 1 & -3 \\ -3 & -2 & 5 \end{pmatrix} \in X$ since $\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Similarly let Y be the subspace of $\mathbb{R}^{2\times 3}$ containing all matrices whose rows add to $0 \in \mathbb{R}^3$. Answer the following questions with reasons.

   (a) (5 points) What is the dimension of X.

   (b) (5 points) What is the dimension of $X + Y$.

4. (10 points) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose that the nullity of T is zero. If $\{x_1, x_2, ..., x_k\}$ is a linearly independent subset of $\mathbb{R}^n$, then show that $\{T(x_1), T(x_2), ..., T(x_k)\}$ is a linearly independent subset of $\mathbb{R}^m$.

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5. (25 points) Let $V = \mathbb{P}^2(\mathbb{R})$, the space of real-valued polynomials of total degree less than or equal to 2. Let $S = \{x, 1-x, 1-x^2\}$ be a set of vectors in V.

   (a) (10 points) Show that S is a basis of V.

   (b) (10 points) Let $T : V \to V$ given by $p(x) \to xp'(x)$. Find the matrix of T with respect to S.

   (c) (5 points) Is T invertible? If not, express its nullspace in terms of S.

6. (15 points) Let A be the matrix

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

   For $u, v \in \mathbb{R}^3$, define $\langle u, v\rangle = u^T A v$, then $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{R}^3$. Under this inner product. find the orthonormal basis $\{w_1, w_2, w_3\}$ by applying Gram-Schmidt process to $\{(1,0,0)^T, (0,1,0)^T, (0,0,1)^T\}$.

7. (10 points) Suppose A is a positive definite symmetric real $n \times n$ matrix and B is a real $m \times n$ matrix such that BB^T is positive definite. Prove that the matrix $B^T(BA^{-1}B^T)^{-1}B$ is symmetric and positive definite.