科目名稱	線性代數	類組代碼	A07.C11
		科目碼	A0702

※本項考試依簡章規定所有考科均「不可」使用計算機。 本科試題共計 6 頁

Single-choice questions

1. (4 points) Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 \\ 5 & 0 & 0 & 1 \end{bmatrix}.$$

Which of the following statements is correct?
(A) $\det(A) = -119$ (B) $\det(A) = 121$ (C) $\det(A) = -121$ (D) $\det(A) = 119$

2. (4 points) Find a to f so that the matrix A is skew-symmetric.

$$A = \begin{bmatrix} 0 & -2 & a & 0 \\ b & 0 & c & 8 \\ 2 & -7 & 0 & d \\ 0 & e & 6 & 0 \end{bmatrix}.$$

(A) $a = -8, b = -6, c = 0, d = 2, e = -2$
(B) $a = -2, b = 2, c = 7, d = -6, e = -8$
(C) $a = 8, b = 6, c = 0, d = -2, e = 2$
(D) $a = 2, b = -2, c = -7, d = 6, e = 8$

3. (4 points) Which of the following matrices can be factorized as $A = LU$, where L is a lower triangular matrix and U is an upper triangular matrix?

(A) $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 1 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ (C) $\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix}$ (D) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix}$

4. (4 points) Let $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}$ be a real matrix such that

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} X = \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 1 & -1 \end{bmatrix}.$$

Which of the following statements is correct?
(A) $x_{12} = 2$ (B) $x_{21} = 5$ (C) $x_{11} = -1$ (D) $x_{32} = 2$

5. (4 points) Let $A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$ and let $\mathrm{adj}(A) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$ denote the classical adjoint of A. Which of the following statements is correct?

(A) $c_{12} = 4$ (B) $c_{21} = 4$ (C) $c_{11} = -1$ (D) $c_{32} = 3$

6. (4 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 200 & 0 & 0 \\ -213 & -10 & 1 & 0 \\ -222 & -12 & 1 & -12 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 & 2 \\ 6 & 4 & 2 & 2 \\ 4 & 2 & 8 & 4 \\ -1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 4 & 3 & 3 & 0 \\ -1 & 2 & 0 & 1 \end{bmatrix}.$$

Find the rank of A.

(A) $\mathrm{rank}(A) = 1$ (B) $\mathrm{rank}(A) = 2$ (C) $\mathrm{rank}(A) = 3$ (D) $\mathrm{rank}(A) = 4$

7. (4 points) Solve k such that the following system

$$\begin{cases} x_1 + 2x_2 - 3x_3 &= 4 \\ 3x_1 - x_2 + 5x_3 &= 2 \\ 4x_1 + x_2 + (k^2 - 14)x_3 &= k + 2 \end{cases}$$

has infinitely many solutions.

(A) $k = 2$ (B) $k = 4$ (C) $k = -4$ (D) $k = 8$

8. (4 points) Let the subspace $\mathbf{W} \subseteq \mathbb{R}^3$ be defined as:

$$\mathbf{W} = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

Which of the following vectors lies in the orthogonal complement $\mathbf{W}^{\perp}$?

(A) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (C) $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ (D) $\begin{bmatrix} -7 \\ -5 \\ 1 \end{bmatrix}$

9. (4 points) Let $u = [-1, 2]$ and $v = [3, -5]$ in $\mathbb{R}^2$, and let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $T(u) = [-2, 1, 0]$ and $T(v) = [5, -7, 1]$. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

be the standard matrix representation of T. Which of the following is TRUE?

(A) $a_{11} = 0$ (B) $a_{12} = 1$ (C) $a_{21} = -2$ (D) $a_{22} = 1$

10. (4 points) Which of the following sets of functions is linearly independent?

(A) $\{e^x, e^{-x}, e^x + e^{-x}\}$

(B) $\{e^x, e^{x+1}, \cos(x), \sin(x)\}$

(C) $\{e^x, e^{-x}, \cosh(x), \sinh(x)\}$

(D) $\{e^x, e^{2x}, e^{3x}\}$

11. (4 points) $T : \mathbb{R}^{2\times3} \to \mathbb{R}^{2\times2}$ defined by $T\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\right) = \begin{bmatrix} 2a_{11} - a_{12} & a_{13} + a_{12} \\ 0 & 0 \end{bmatrix}$.
Find $\text{rank}(T)$.

(A) $\text{rank}(T) = 1$ (B) $\text{rank}(T) = 2$ (C) $\text{rank}(T) = 3$ (D) $\text{rank}(T) = 4$

12. (4 points) Let

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Classify matrices A, B, C, and D according to the categories: (1) Positive Definite, (2) Negative Definite, (3) Positive Semi-Definite, (4) Negative Semi-Definite. Which of the following gives the correct order of matrices from category (1) to (4)?

(A) BACD (B) ADCB (C) ADBC (D) DABC

13. (4 points) Which of the following statements is FALSE about the algebraic and geometric multiplicities of the eigenvalues of A?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{bmatrix}.$$

(A) The algebraic multiplicity of the eigenvalue -1 is 1.

(B) The algebraic multiplicity of the eigenvalue 2 is 2.

(C) The geometric multiplicity of the eigenvalue -1 is 1.

(D) The geometric multiplicity of the eigenvalue 2 is 2.

14. (4 points) $W_1 = \left\{\begin{bmatrix} a & b \\ c & a \end{bmatrix}, a, b, c \in \mathbb{R}\right\}$, $W_2 = \left\{\begin{bmatrix} 0 & a \\ -a & b \end{bmatrix}, a, b \in \mathbb{R}\right\}$. Find the dimension of $W_1 \cap W_2$.

(A) $\dim(W_1 \cap W_2) = 1$

(B) $\dim(W_1 \cap W_2) = 2$

(C) $\dim(W_1 \cap W_2) = 3$

(D) $\dim(W_1 \cap W_2) = 4$

15. (4 points) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Compute e^A.

(A) $e^A = \begin{bmatrix} \frac{e^2}{2} & \frac{e}{2} \\ \frac{e}{2} & \frac{e^2}{2} \end{bmatrix}$ (B) $e^A = \begin{bmatrix} \frac{e}{2} & 0 \\ 0 & \frac{e^3}{2} \end{bmatrix}$ (C) $e^A = \begin{bmatrix} \frac{e+e^3}{2} & \frac{-e+e^3}{2} \\ \frac{-e+e^3}{2} & \frac{e+e^3}{2} \end{bmatrix}$ (D) $e^A = \begin{bmatrix} \frac{e+e^2}{2} & \frac{-e+e^2}{2} \\ \frac{-e+e^2}{2} & \frac{e+e^2}{2} \end{bmatrix}$

16. (4 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a function. Which of the following defines a linear transformation?

(A) $T(a_1, a_2) = (1,\ a_1)$

(B) $T(a_1, a_2) = (a_1,\ a_2^2)$

(C) $T(a_1, a_2) = (a_1 + 1,\ a_2)$

(D) $T(a_1, a_2) = (a_1 \cos\theta - a_2 \sin\theta,\ a_1 \sin\theta + a_2 \cos\theta)$ for a fixed angle θ

17. (4 points) Which of the following statements is FALSE?

(A) S is a basis for a vector space V if and only if S is a minimal spanning set.

(B) V is a finite-dimensional vector space ($\dim(V) < \infty$), and W is a subspace of V. If $\dim(V) = \dim(W)$, then $W = V$.

(C) Let V and V' be vector spaces having the same finite dimension, and let $T : V \to V'$ be a linear transformation. Then T is one-to-one if and only if range$(T) = V'$

(D) Let V and V' be vector spaces of dimensions n and m, respectively. A linear transformation $T : V \to V'$ is invertible if and only if $m = n$.

Multiple-choice questions

18. (4 points) Which of the following matrices are unitary?

(A) $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$ (B) $\begin{bmatrix} 1 & i \\ 0 & i \end{bmatrix}$ (C) $\frac{1}{3} \begin{bmatrix} 2 & -2+i \\ 2+i & 2 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

19. (4 points) Considering the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{bmatrix}.$$

The eigenvalues of the matrix A are

(A) -1 (B) 1 (C) 6 (D) 7 (E) 2

20. (4 points) Which of the following statements are correct?

(A) A linear system with fewer equations than unknowns may have no solution.

(B) Every linear system with the same number of equations as unknowns has a unique solution.

(C) A linear system with coefficient matrix A has an infinite number of solutions if and only if A can be row-reduced to an echelon matrix that includes some column containing no pivot.

(D) If $[A|b]$ and $[B|c]$ are row-equivalent partitioned matrices, the linear systems $Ax = b$ and $Bx = c$ have the same solution set.

(E) A linear system with a square coefficient matrix A has a unique solution if and only if A is row equivalent to the identity matrix.

21. (4 points) Which of the following statements are NOT correct?

(A) Let A be a real $n \times n$ matrix. If A^2 is invertible, then $A^\top$ and A^3 are invertible.

(B) If A and B are invertible, then so is $A + B$, and $(A + B)^{-1} = A^{-1} + B^{-1}$.

(C) Let A be a real $m \times n$ matrix and B a real $n \times m$ matrix. Then $\text{trace}(AB) = \text{trace}(BA)$.

(D) Let A be a real $m \times n$ matrix and B a real $n \times m$ matrix. Then $\det(AB) = \det(BA)$.

(E) Let A and B be real $n \times n$ matrices. Then $AB = BA$.

22. (4 points) Which of the following statements are NOT correct?

(A) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable.

(B) Two distinct eigenvectors corresponding to the same eigenvalue are always linearly independent.

(C) If λ is an eigenvalue of a linear operator T, then each vector in the eigenspace E_λ is an eigenvector of T.

(D) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of the corresponding eigenspace E_λ.

(E) If A is diagonalizable, then $A^\top$ is also diagonalizable.

23. (4 points) Which of the following statements are NOT correct?

(A) If S is linearly independent and generates V, each vector in V can be expressed uniquely as a linear combination of vectors in S.

(B) Every vector space has at least two distinct subspaces.

(C) No vector is its own additive inverse.

(D) All vector spaces having a basis are finitely generated.

(E) Any two bases in a finite-dimensional vector space V have the same number of elements.

24. (4 points) Let W_1 and W_2 be subspaces of a finite-dimensional vector space V. Let $\oplus$ denote the direct sum. Which of the following statements are correct?

(A) $W_1 \cap W_2$ is a subspace of V.

(B) $W_1 \cup W_2$ is a subspace of V.

(C) $W_1 + W_2$ is a subspace of V.

(D) If $V = W_1 \oplus W_2$, and β_1 and β_2 are bases for W_1 and W_2, respectively, then $\beta_1 \cap \beta_2 = \emptyset$, and $\beta_1 \cup \beta_2$ is a basis for V.

(E) If $W_1 \oplus W_2 = V$, then the dimension $\dim(V) = \dim(W_1) + \dim(W_2)$.

25. (4 points) Which of the following statements are correct?

(A) If Q is orthogonal, then $\det(Q) = \pm 1$.

(B) Let A be a real $n \times n$ matrix. Then A is symmetric if and only if A is orthogonally equivalent to a real diagonal matrix.

(C) Let $A \in \mathbb{R}^{n \times n}$ be a matrix whose characteristic polynomial splits over $\mathbb{R}$. Then A is orthogonally equivalent to a real upper triangular matrix.

(D) Let T be a self-adjoint (Hermitian) operator on a finite-dimensional inner product space V. Then every eigenvalue of T is positive.

(E) Let T be a self-adjoint (Hermitian) operator on a finite-dimensional inner product space V. Then every eigenvalue of T is negative.